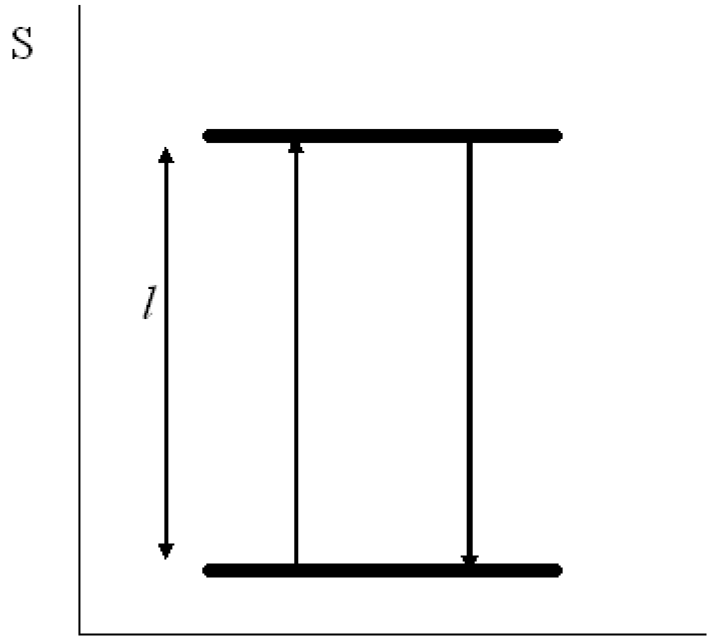
Activity 1 - Time Dilation

In relativity different observers can (correctly) disagree on the time between events.

Put another way; “moving clocks tick slow” - What does this mean?

- Take a spaceship flying to Alpha Centauri (the sun’s nearest neighbouring star) at half the speed of light, as measured by observers on Earth.
- When the astronauts look at their clocks on their spaceship, they see time ticking by normally: they share the same inertial frame: the spaceship.
- When the observers back on Earth look at their clocks on Earth, they also see time ticking normally: the Earth is their inertial frame.
- However, say that every time the clocks on the spaceship tick, the spaceship sends out a light pulse back to earth.
- The observers on the Earth receive the pulses arriving every 1.15 ticks of their clocks on Earth! i.e. the astronauts see the pulses emitted every second but observers receive them every 1.15s.
- This is *time dilation*, and we will now prove this result.

In an inertial frame, S, we have a light clock, where each time the light pulse touches the bottom mirror we count a tick on the clock.

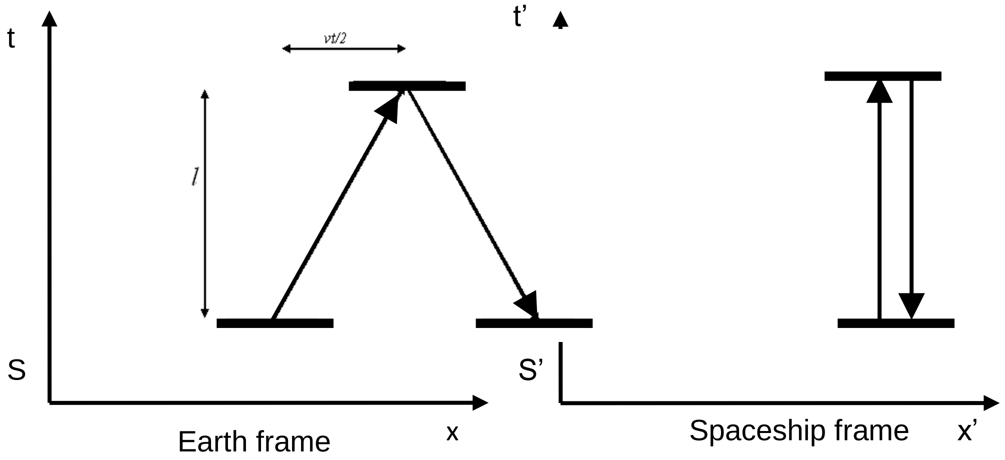


Since the speed of light is c, we have a tick every $T = \frac{2l}{c}$.

Remember, this is an inertial frame. This means that the frame is not accelerating - it could be moving at a large fraction of the speed of light, but as long as the speed is constant, sitting in the frame with the clock, we cannot tell whether we are moving in any absolute terms. Indeed, in relativity there are no absolute frames of reference - we can only measure speeds relative to other things.

- Now, we put our light clock in a spaceship and send it out at a speed v, while we remain in our frame S.
- The spaceship defines a new inertial frame, S'.
- In S', the light clock looks exactly as it does above, because in S' the clock is still stationary.
- In our frame, S, the clock is moving at speed v.

So how do we, stationary in our frame S, see the clock in S' which is moving at v?



If the above diagram is not obvious, think about the simultaneity of events discussed in the links above. We can call the light pulse leaving and arriving at mirrors "events": points in both space and time. In S' the light pulse hits a mirror and is reflected back. If the mirror and pulse weren't at the same point at the same time the pulse would just keep going as it was. Whichever frame we are in, all observers have to agree that the light pulse hits the mirror.

We will call the time as measured by the observers in S', t', and the time measured by observers in S, t.

So, in S', since the clock is stationary, one "tick" of the clock (light pulse travels up and back down to the bottom mirror) takes time:

$t' = \frac{2l}{c}$,

just as before.

To find the time between ticks in S, we just need a little geometry. Using Pythagoras, going from the bottom mirror to the top, the light travels:

$$d = \sqrt{l^2 + \left(\frac{vt}{2}\right)^2}$$

so:

$t = \frac{d}{c} = \frac{2\sqrt{l^2+\left(\frac{vt}{2}\right)^2}}{c}$.

It is clear that $t \neq t''$! We want to know how the times are related, so we rearrange the equation for t and substitute for l:

$\frac{t^2}{4}\left(1 - \frac{v^2}{c^2}\right) = \frac{l^2}{c^2}$,

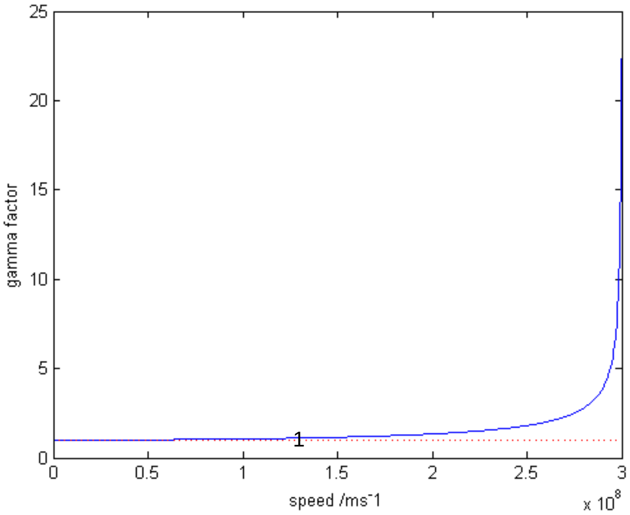
$t^2\left(1 - \frac{v^2}{c^2}\right) = t'^2$,

$t = \gamma t'$,

where $\gamma \equiv \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$.

Which is mathematical relation for time dilation! Make sure you work through the algebra and are comfortable with each step.

γ is one of the most important factors in special relativity. If you look at the case where v is small, you can see γ tends to 1. What about at high speeds? Sketch γ:



Key Point: we have written t and t', suggesting specific times, when as should be clear from the working, these are really time intervals. They are the times taken for the light pulse to go up and back down inside our clock.

We have used a light clock, but what if we used an atomic clock, or some as yet uninvented clock. How do we know this time dilation result still holds? Very simply, if it did not, then we could compare our new clock to the light clock, and see if we were moving at a constant velocity, without any other reference frame! This contradicts the earlier statements of the theory; that all inertial frames are equivalent.

Worked Example:

Imagine we had a spaceship which could instantaneously accelerate to whatever speed we wanted (below the speed of light). Suppose we also found a star 10 000 light years (lyr) away which we wanted to visit. We decide that we want our journey to take 10 years (in the frame of the spaceship).

How fast do we need to travel, and how long does it take for us to reach the star in the Earth's frame?

Solution:

- We know that the time interval in the spaceship, t', and on the Earth, t, are related by $\mathrm{t} = \gamma \mathrm{t}'$.
  (remembering that clocks tick slower on the moving object)
- We also know that in the Earth frame the spaceship will take time $\mathrm{t} = \frac{\mathrm{d}}{\mathrm{v}}$ to reach the star, where $\mathrm{d} = 10{,}000\ \mathrm{lyr}$ and v is as yet unknown.
- Expand γ, and the two equations can be combined to give:

$$\mathrm{t}' = \frac{\mathrm{d}}{\mathrm{v}}\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}.$$

- Square both sides and rearrange:

$$\frac{\mathrm{v}^2}{\mathrm{c}^2} = \frac{\mathrm{d}^2}{\mathrm{d}^2 + \mathrm{c}^2\mathrm{t}'^2}$$

- Now we could convert everything in SI units and back again, or we could work in units of years and light years. Clearly $\mathrm{c} = 1\ \mathrm{lyr\ yr^{-1}}$.Substituting in the values from above:

$$\frac{v}{c} = \sqrt{\frac{(10{,}000\ lyr)^2}{(10{,}000\ lyr)^2\ +\ (1\ lyr\ yr^{-1})^2(10\ yr)^2}} = 0.999{,}999{,}5$$

- Where the answer is expressed as a fraction of the speed of light. Needless to say, this is very fast!
- Still working in years and light years we can now easily find the time in the Earth frame, t:

$$t = \frac{d}{v} = \frac{10{,}000\ \mathrm{lyr}}{0.999{,}999{,}5\ \mathrm{lyr\ yr^{-1}}} = 10{,}000.005\ \mathrm{yr}$$

This is in fact the starting point of the Twin Paradox mentioned in many of the links in this document. Again we emphasise that the 10 years on the spaceship are not just an illusion: people would really only age 10 years while 10,000 years passed on Earth.

While the idea of an instantaneously accelerated spaceship is used here to simplify the maths, the same principle holds for a ship which accelerated until it was halfway to the star, then decelerated the rest of the way. This would allow a much more comfortable and achievable acceleration.